21.02.2026

Theorem Let ~~E~~ X and Y be two metric spaces and $f : X \to Y$ be a mapping. Then f is continuous iff $f^{-1}(G)$ is open in X whenever G is open in Y.

Proof **Necessary part** Let $f : X \to Y$ be continuous. Let G be an open set in Y. We must show that $f^{-1}(G)$ is open in X.

If G is empty $\Rightarrow f^{-1}(G)$ is open.

So, let G be non-empty.

Let $x \in f^{-1}(G)$. $\Rightarrow f(x) \in G$.

But G is open $\Rightarrow \exists$ an open sphere $S_\epsilon(f(x))$ with centre on $f(x)$ contained in G. ~~such~~

Since f is continuous,

$\Rightarrow \exists$ an open sphere $S_\delta(x)$ such that

$f(S_\delta(x)) \subseteq S_\epsilon(f(x))$.

$\because S_\epsilon(f(x)) \subseteq G \Rightarrow f(S_\delta(x)) \subseteq G$

$\Rightarrow S_\delta(x) \subseteq f^{-1}(G)$. $\Rightarrow S_\delta(x)$ is an open sphere with centre on x and contained in $f^{-1}(G)$.

$\Rightarrow f^{-1}(G)$ is open in X.

Sufficient part   Given that

G is open $\Rightarrow f^{-1}(G)$.

we have to prove that f is continuous.

we show that f is continuous at an arbitrary point x in X.

Let $S_\epsilon(f(x))$ be an open sphere with centre on $f(x)$.

This open sphere is an open set.

$\Rightarrow$ its inverse image is an open set which contains x.

$\Rightarrow \exists$ an open sphere $S_\delta(x)$ which is contained ~~in $S_\epsilon(f(x))$~~ in this inverse image.

$\Rightarrow f(S_\delta(x))$ is contained in $S_\epsilon(f(x))$

$\Rightarrow f$ is continuous at x.

Since x is arbitrary point in X

$\Rightarrow f$ is continuous.